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SINGLE VS. MULTIPLE ASSIGNMENT IN HUB-AND-SPOKE NETWORK: A TOTAL COST COMPARISON

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Summary: The paper deals with total costs comparison of two hub location models – single and multiple allocation. The impact of inter-hub discount factor is examined on three different numerical examples.

Key words: Hub-and-Spoke, location, single allocation, multiple allocation

1. INTRODUCTION

Hub-and-Spoke networks are used in various types of network systems (e.g. freight railway transportation, airline transportation, postal delivery systems, and telecommunications). Their principle can be described as follows: There exist many origins, many destinations and a number of hubs (i.e. transshipment facilities in case of transport systems). All the transport flow must travel from an origin to a destination via either one or two hubs. The flow is then concentrated on the inter-hub links and it brings transportation economies of scale (expressed by a discount factor).

There exist several types of Hub-and-Spoke networks; this article deals with two of them: the single allocation and the multiple allocation Hub-and-Spoke networks. Both types are considered to be uncapacitated and with given number of hubs. These two types of network systems will be compared with regard to total cost and sensitivity of solutions to the discount factor.

The paper has several goals:

- to illustrate the impact of discount factor on solution of hub location problems;
- to illustrate the cost difference between single and multiple allocation;
- to analyze and compare the impact of the changes in hub location model on different data sets.

2. MATHEMATICAL FORMULATION OF BOTH PROBLEMS

2.1 Inter-hub economies of scale and total costs calculation explained

The entire transportation process from origin to destination can be divided into three parts: a collection part (origin-hub), an inter-hub part (hub-hub) and a distribution part (hub-destination). The consolidation of shipments on inter-hub links provides the possibility to use large vehicles and to increase the unit transportation costs (in comparison to collection and distribution). The simplest way how to express these economies of scales is to calculate the unit transportation cost from node i to node j using hubs k, l by (1).

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$$c_{iklj} = d_{ik} + \alpha * d_{kl} + d_{lj} \tag{1}$$

where:

- d_{ik} ………… length of collection part,
- α ………… inter-hub discount factor ($0 \leq \alpha \leq 1$),
- d_{kl} ………… length of inter-hub part,
- d_{lj} ………… length of distribution part.

It is possible to establish collection and distribution factors as well - if the collection and distribution unit costs differ.

Notice that the real unit transportation costs (in terms of *vehicle operation cost per unit of distance* divided by *vehicle capacity*) are not considered. The reason is simple – for solution of the problem of optimal location and allocation is enough to know the proportion of real unit cost. For example, if the real unit transportation cost of large vehicles (used on inter-hub links) equals to L monetary units and real unit transportation cost of smaller vehicles (used for visiting origins and destinations) is equal to S monetary units, then the discount factor $\alpha = L/S$; the collection and distribution factors are in fact equal to 1.

The total costs in monetary units can be than calculated as total transportation costs (i.e. the value of objective function (2) or (7)) multiplied by S. This is important especially when including fixed costs to operate and establish hubs into consideration.

2.2 Single Assignment Hub-and-Spoke problem

In the single allocation model all nodes are restricted to interact with just one hub (see Figure 1). The uncapacitated Hub-and-Spoke problem with given number of hubs (this number is equal to p) and single allocation of nodes to hubs can be stated as follows:

$$\text{minimize } \sum_i \sum_j b_{ij} \left(\sum_k d_{ik}\, h_{ik} + \sum_k \sum_l \alpha\, d_{kl}\, h_{ik}\, h_{jl} + \sum_l d_{jl}\, h_{jl} \right) \tag{2}$$

subject to:

$$\sum_k h_{kk} = p \tag{3}$$

$$\sum_k h_{ik} = 1 \qquad \forall\ i \in V \tag{4}$$

$$h_{ik} \leq h_{kk} \qquad \forall\ i, k \in V \tag{5}$$

$$h_{ik} \in \{0,1\} \qquad \forall\ i, k \in V \tag{6}$$

where:

- b_{ij} ………… flow size from node i to node j,
- h_{ik} ………… decision variable; $h_{ik} = 1$ if node i is allocated to hub k,
- h_{jl} ………… decision variable; $h_{jl} = 1$ if node j is allocated to hub l,
- h_{kk} ………… decision variable; $h_{kk} = 1$ if node k is hub,
- V ………… set of all nodes of given network.

The objective (2) represents total transportation costs. The constraint (3) specifies the number of hubs to be opened; the constraints (4) guarantee single allocation. The constraints (5) ensure, that node k is a hub before a node can be allocated to it. The formulated problem is NP-hard.

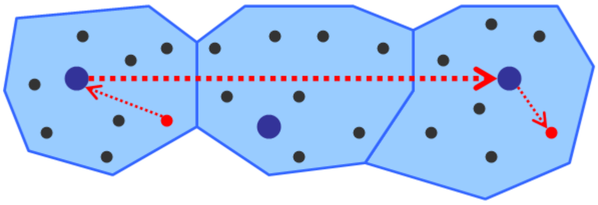

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Fig. 1 – Single assignment model example

2.3 Multiple Assignment Hub-and-Spoke problem

The multiple allocation model allows each origin-destination pair of nodes to select the most effective route (i.e. route with minimum costs) through the hubs. The advantage of this model is the costs reduction (see Figure 2); the disadvantage is then more complicated collection and distribution – each node can be operated by more than one hub.

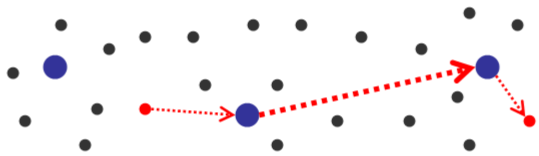

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Fig. 2 – Multiple assignment model example

The uncapacitated Hub-and-Spoke problem with given number of hubs and multiple allocation of nodes to hubs can be stated as follows:

$$\text{minimize } \sum_{i}\sum_{j}\sum_{k}\sum_{l} b_{ij}\, c_{ijkl}\, X_{ijkl} \tag{7}$$

subject to:

$$\sum_{k}\sum_{l} X_{ijkl} = 1 \qquad \forall\, i, j \in V \tag{8}$$

$$\sum_{k} Y_k = p \tag{9}$$

$$X_{ijkl} \le Y_k \qquad \forall\ i, j, k, l \in V \tag{10}$$

$$X_{ijkl} \le Y_l \qquad \forall\ i, j, k, l \in V \tag{11}$$

$$X_{ijkl} \in \{0,1\} \qquad \forall\ i, j, k, l \in V \tag{12}$$

$$Y_k \in \{0,1\} \qquad \forall\ k \in V \tag{13}$$

where:

X_{ijkl} ………… decision variable; $X_{ijkl} = 1$ if the origin-destination pair i, j used hub k and l (in this order),

Y_k ………… decision variable; $Y_k = 1$ if node k is hub,

the other variables have the same meaning as in expressions (1) – (6).

The unit transportation costs can be calculated by (1); the objective (7) then expresses the total transportation cost. The constraints (8) guarantee that the flow between every origin-destination pair will be routed via some pair of hubs. The constraint (9) specifies the number of hubs to be opened. The constraints (10) and (11) ensure that every origin-destination flow is routed only via nodes which are hubs. The formulated problem is again NP-hard.

3. COST COMPARISON ILLUSTRATED

Let's consider an example of Hub-and-Spoke network as given in figure 3. There are 3 hub nodes – node 1, 2 and 4. To keep it simple, the OD matrix contains only two flows: 40 units from node 3 to node 6 and 5 units from node 3 to node 7.

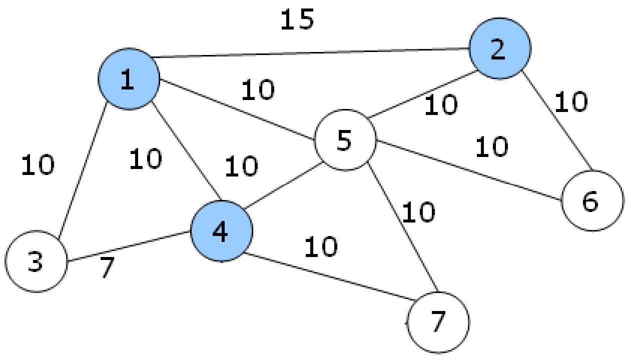

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Fig. 3 – Simple network example

The assignment of nodes 6 and 7 appears quite obvious: node 6 will be allocated to hub 2 and node 7 will be allocated to node 4. It will be shown that it is not always optimal to assign nodes to the closest hub; these allocations are assumed to be given in this example anyway. What about the assignment of node 3?

Let's consider subsequent allocation to hub 4 and then to hub 1 for 3 different values of discount factor $\alpha = 1$ (no economies of scale), $\alpha = 0.9$ and $\alpha = 0.6$. The results are showed in Tables 1 and 2. The OD cost values stand for unit transportation cost (i.e. where inter-hub cost is discounted by α) multiplied by corresponding flow size.

Table 1 – Results for node 3 allocated to node 4

Route	Travel distance	Unit cost $\alpha = 1$	OD cost $\alpha = 1$	Unit cost $\alpha = 0.9$	OD cost $\alpha = 0.9$	Unit cost $\alpha = 0.6$	OD cost $\alpha = 0.6$
3–4–2–6	37	37	1 480	35	1 400	29	1 160
3–4–7	17	17	85	17	85	17	85
			∑ 1 565		∑ 1 485		∑ 1 245

Source: Author

Table 2 – Results for node 3 allocated to node 4

Route	Travel distance	Unit cost α = 1	OD cost α = 1	Unit cost α = 0.9	OD cost α = 0.9	Unit cost α = 0.6	OD cost α = 0.6
3–1–2–6	35	35	1 400	33.5	1 340	29	1 160
3–1–4–7	30	30	150	29	145	26	130
			∑ 1 550		∑ 1 485		∑ 1 290

Source: Author

In case of single allocation the node 3 should be assigned to hub 1 for values of $\alpha > 0.9$. For values of $\alpha < 0.9$ is better to allocate node 3 to node 4. In case of multiple allocation it is advisable to assign node 3 to hub 1 for values of $\alpha > 0.6$ and to node 1 for $\alpha < 0.6$ in case of transportation from node 3 to node 6. In case of transportation from node 3 to node 7 should be always used only hub 4. The optimum solutions can be found in Table 3.

Table 3 – Solutions for both allocation types

Discount factor	Optimum solution for single allocation	Optimum solution for multiple allocation
α = 1	3–1–2–6; 3–1–2–6 (total costs = 1 550)	3–1–2–6; 3–4–7 (total costs = 1 4 85)
α = 0.9	3–1–2–6; 3–1–2–6 or 3–4–2–6; 3–4–7 (total costs = 1 485)	3–1–2–6; 3–4–7 (total costs = 1 425)
α = 0.6	3–4–2–6; 3–4–7 (total costs = 1 245)	3–1–2–6; 3–4–7 or 3–4–2–6; 3–4–7 (total costs = 1 245)

Source: Author

It is possible to make some conclusions: It is obvious that the value of discount factor α influences the optimal assignment of nodes to hubs. As α decreases, the length of collection and distribution part of travel distance plays a larger role. Thus, the closest hub does not necessary means the optimal hub – this is valid for single allocation as well as for multiple allocation. For decreasing values of α, the cost difference between single and multiple allocation becomes smaller and the solutions are more similar. In the case when $\alpha = 0$ (i.e. there are no costs for inter-hub transport) both solutions should be the same. The reason is obvious – if there are no costs between hubs, each node should be allocated to closest hub and multiple allocations are not needed.

4. COMPUTATIONAL STUDY - REAL LIFE EXAMPLES

4.1 Used algorithms

All computations were done using the software HubLoc [7] which was created by author. The single allocation algorithm is based on genetic algorithm as described in [3]. This algorithm was slightly modified to allow multiple allocation as well (though it suffers by performance issues).

Both types of algorithms were tested and verified on standard datasets available from [5]. Notice that the results come from metaheuristic algorithms – all the solutions are suboptimal in general.

4.2 Used data

Three different data sets were used. First data set CAB (Civil Aeronautics Board) is based on airline passenger travel between 25 U.S. cities in 1970. Second data set which is called AP (Australian Post) relates to post shipments delivery between 200 Australian cities. These data were reduced to 50 nodes using clustering as described in [2]. Both data sets are commonly used for benchmarking algorithms dedicated for solving of Hub-and-Spoke problems and can be obtained from [5]. The third data set CR (Czech Railways) contains data about freights transportation flows between 50 Czech railway districts in 2007 [6].

4.3 Comparison of overall results

Both models - single and multiple allocation - were solved using HubLoc software, various values of discount factor α and fixed number of hubs (3 and 4). Results are shown in Tables 4 – 6. Column TCS stands for total costs of single allocation; TCM means total costs of multiple allocation; TC% is calculated as TCM / TCS · 100 with the same value of α and the same number of hubs; NMA stands for number of nodes which use multiple allocation. Such comparison was done in [6] using CAB dataset; one of the goals of this paper is to compare the results with other data sets.

Table 4 – CAB data set (25 nodes) solutions

	3 HUB model				4 HUB model			
α	**TCS ($\cdot 10^9$)**	**TCM ($\cdot 10^9$)**	**TC%**	**NMA**	**TCS ($\cdot 10^9$)**	**TCM ($\cdot 10^9$)**	**TC%**	**NMA**
0	5,363	5,363	100,0	0	3,938	3,938	100,0	0
0,1	5,972	5,930	99,3	2	4,669	4,640	99,4	1
0,2	6,553	6,430	98,1	5	5,377	5,282	98,2	7
0,3	7,135	6,902	96,7	8	6,078	5,886	96,8	7
0,4	7,701	7,341	95,3	9	6,725	6,443	95,8	9
0,5	8,348	7,742	92,7	12	7,373	6,954	94,3	12
0,6	8,898	8,106	91,1	14	8,021	7,399	92,3	13
0,7	9,362	8,425	90,0	18	8,663	7,788	89,9	17
0,8	9,960	8,711	87,5	19	9,289	8,128	87,5	19
0,9	10,489	8,940	85,2	21	9,914	8,407	84,8	20
1	10,943	9,071	82,9	24	10,390	8,597	82,7	24

Source: Author

Table 5 – AP data set (50 nodes) solutions

	3 HUB model				4 HUB model			
α	**TCS ($\cdot 10^7$)**	**TCM ($\cdot 10^7$)**	**TC%**	**NMA**	**TCS ($\cdot 10^7$)**	**TCM ($\cdot 10^7$)**	**TC%**	**NMA**
0	5,474	5,474	100,0	0	4,725	4,725	100,0	0
0,1	5,822	5,798	99,6	5	5,153	5,140	99,7	4
0,2	6,140	6,092	99,2	8	5,500	5,448	99,1	10
0,3	6,447	6,351	98,5	11	5,844	5,749	98,4	13
0,4	6,752	6,584	97,5	16	6,185	6,012	97,2	17
0,5	7,056	6,777	96,0	23	6,501	6,241	96,0	21
0,6	7,361	6,936	94,2	27	6,862	6,445	93,9	26
0,7	7,682	7,070	92,0	31	7,183	6,623	92,2	30
0,8	7,936	7,177	90,4	36	7,485	6,770	90,4	36
0,9	8,189	7,314	89,3	41	7,784	6,874	88,3	43
1	8,438	7,283	86,3	49	8,086	6,925	85,6	49

Source: Author

Table 6 – CR data set (50 nodes) solutions

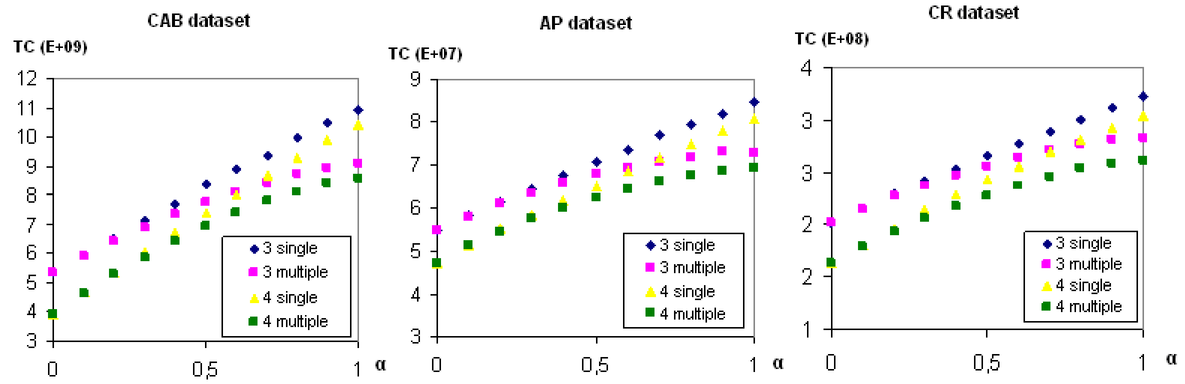
	3 HUB model				4 HUB model			
α	TCS (· 10^8)	TCM (· 10^8)	TC%	NMA	TCS (· 10^8)	TCM (· 10^8)	TC%	NMA
0	2,009	2,009	100,0	0	1,631	1,631	100,0	0
0,1	2,153	2,146	99,7	6	1,797	1,786	99,4	8
0,2	2,295	2,266	98,7	7	1,961	1,923	98,1	12
0,3	2,419	2,378	98,3	11	2,142	2,051	95,8	17
0,4	2,535	2,470	97,4	18	2,287	2,165	94,7	21
0,5	2,650	2,556	96,5	18	2,431	2,269	93,3	20
0,6	2,765	2,639	95,4	22	2,552	2,370	92,9	21
0,7	2,881	2,716	94,3	27	2,700	2,460	91,1	27
0,8	2,996	2,772	92,5	29	2,815	2,543	90,3	41
0,9	3,111	2,810	90,3	33	2,930	2,586	88,3	44
1	3,227	2,832	87,8	48	3,043	2,615	85,9	49

Source: Author

The obtained results can be summarized as follows:

- For a given number of hubs the total cost for the multiple assignment is always less or equal than in case of multiple assignment. Only for very small value of discount factor (close to zero) are both costs the same. As α increases, the difference increases.
- For small values of α both models have tendency to use nearest hubs, with increasing value of α it is no always optimal for nodes to use the nearest hub – it is expected not only in the case of multiple allocation, but also in the case of single allocation when it depends on layout of transportation flows.
- As α increases, the number of multiple assignments increases. It reaches 0 % for α = 0 and almost 100 % when α = 1. Multiple allocations occurred in all cases even for very small values of α.
- As α increases, the inter-hub distances play more important role in total transportation costs. It causes changes in hub locations – hubs move closer together.

Although the computational experiment was done using 3 different data sets, the obtained results are very similar in response on change of α and number of hubs. This similarity is clearly visible on Figure 4.



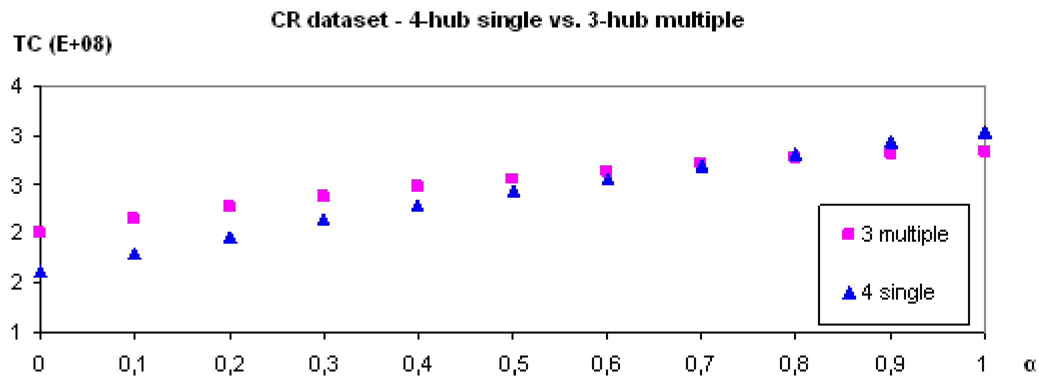
Source: Author

Fig. 4 – Comparison of results for particular datasets

4.4 More hubs and single allocation vs. fewer hubs and multiple allocation

Obtained results also show following fact: under certain circumstances can be fewer hubs with multiple allocation more effective than more hubs with single allocation. For smaller and moderate values of discount factor α is cheaper the 4 hub single assignment model, but for higher values of α (the margin is about 0.7 in case of CAB and AP data sets; in case of CR – see Figure 5 - is about 0.8) the 3 hub multiple assignment model takes this advantage. This is because of the increase in costs to operate on inter-hub links – under such circumstances is the multiple allocation (which means routes shortening in general) more advantageous. Notice that the fixed costs to operate and establish one more hub are not considered – the actual costs advantage can be in fact even higher.

It is possible to make this conclusion: if the discount on inter-hub links is small, it is better to allow multiple assignments (it means to add collection / distribution links) than to increase the number of hubs.



Source: Author

Fig. 5 – Comparison of total cost for CR data set with 3 hubs and multiple allocation vs. 4 hubs and single allocation

5. CONCLUSIONS

The main goal of this paper is a comparison of total transportation costs in case of single and multiple assignment of nodes to hubs. It was verified that the multiple allocation can give significant cost advantage – particularly when the inter-hub economies of scale are relatively small. The obtained conclusions are not new; but it can make the concept of hub location model more intuitive to wider public. Concerning to the comparison of the three data sets - the impact of changes in allocation type and in the discount factor height was very similar.

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